ARL-MR-0900 ● SEP 2015

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Application of Force and Energy Approaches to the Problem of a One-Dimensional, Fully Connected, Nonlinear-Spring Lattice Structure

by Steven B Segletes

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REPORT DOCUMENTATION PAGE

1. REPORT DATE *(DD-MM-YYYY)* September 2015	2. REPORT TYPE Final	3. DATES COVERED (From - To) June 2015

4. TITLE AND SUBTITLE Application of Force and Energy Approaches to the Problem of a One-Dimensional, Fully Connected, Nonlinear-Spring Lattice Structure	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S) Steven B Segletes	5d. PROJECT NUMBER AH80
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) U.S. Army Research Laboratory ATTN: RDRL-WMP-C Aberdeen Proving Ground, MD 21005-5066	8. PERFORMING ORGANIZATION REPORT NUMBER ARL-MR-0900

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES
author's email: <steven.b.segletes.civ@mail.mil>

14. ABSTRACT
In this report, force and energy methods are applied to the problem of 1-dimensional, fully connected, nonlinear-spring lattices. The report confirms that energy and force methods produce equivalent results, even when nonlinear, non-local effects come into play. Demonstrating this compatibility is slightly complicated for the problem of fully connected lattices, because the boundary conditions of simple loading and uniform spacing are, in general, incompatible. In this report, both boundary conditions are studied separately and it is indeed shown that the force approach (involving free-body diagrams) and the energy approach (involving potential spring energy) produce compatible measures of lattice force, if the conceptualization is properly formulated. Two interesting revelations ensued from employing an energy approach: 1) the lattice force (a derivative quantity) is not equal to the net applied external load, but rather it equals the average internal force across the lattice; and 2) the global lattice force, being a sum of internodal forces, nonetheless requires a multiplier on each local internodal force proportional to the internodal separation. The multiplier arises from an application of the chain rule, when taking the spatial derivative of energy.

15. SUBJECT TERMS
Force, energy, lattice, nonlinear, fully connected, nearest neighbors, free-body diagram

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON Steven B Segletes
a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified	UU	46	19b. TELEPHONE NUMBER (Include area code) 410-278-6010

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1. Introduction

The application of force and energy approaches to solve problems in mechanics have competed side by side for 160 years. Historically, the force balancing approach preceded the development of energy methods that are based on conservation principles. Because of the relation that links a change in energy (E) to the action of a mechanical force (F) acting through a distance, $dE = -F\,dx$, we are content that the 2 methods produce equivalent results when applied to the same problem. The method of choice depends on the particular problem to be addressed and/or the proficiency of the practitioner with the various methods. On one hand, the force method can be conceptually more straightforward; however, an energy approach, once properly conceptualized, often proves more efficient in practice.

The tension between force and energy approaches, however, has had an evolving history. Leaders in the 19th century physics community[1] debated long and hard on notions of conservation, and the relationship between force approaches and the newer energy approaches. A prevailing concept of force "conservation," championed by Faraday and taken to mean the "indestructibility and transformability of natural agents," was considered mathematically imprecise and ambiguous by others. Rankine, on the other hand, had posited the law of energy conservation in 1853. By the end of the decade, physicists such as Helmholtz, Thomson, Rankine, and Maxwell wished to excise the force-conservation concept from the technical vocabulary.

In this report, we examine 1-dimensional (1-D) lattices from both a force and energy approach. The goal is merely to confirm the compatibility of the 2 methods as applied to the mechanical interactions of 1-D lattices. Specifically, in the energy approach, energy is directly tabulated, while force is treated as a derivative quantity, based on the relation $F = -dE/dx$. It is determined that the application of this relation both macroscopically (*i.e.*, where F, E, x are the lattice's external force, total energy, and body dimension, respectively) and internodally (*i.e.*, where F, E, x are the internodal force, energy, and separation, respectively) are compatible with the force approach to describing the lattice equilibrium.

The analysis of mass nodes interconnected with springs is a well-established field and forms the basis for the finite-element method (FEM).[2,3] The use of both force and energy methods in FEM has been employed, historically. When generalizing

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the approach to large systems, the lattice connectivity is typically expressed by way of a stiffness matrix, to which the methods of linear algebra are applied for a solution. However, the traditional mechanical analysis of lattices (and this applies to the FEM as well) is often accompanied by approximations of the physical system. One common simplification is the "harmonic assumption," in which the analysis assumes the lattice springs are linear (*i.e.*, stiffness is independent of extension). Another typical assumption is that of a simply connected lattice, in which each node of the lattice interacts with only its nearest neighbors.

In this report, however, we choose not to limit ourselves in the manner described above. This analysis considers fully connected 1-D lattices, in which each node of the lattice experiences a mechanical interaction with every other node in the lattice. Thus, non-local node interactions are considered. In addition, no harmonic assumption is made, in which the spring constants are invariant with extension. In fact, no particular spring-force relationship is assumed at all, except for the purposes of illustration. Our only assumption, for the purpose of these derivations, is that the component spring force $f(x)$ is knowable, from which the necessary external force(s) may be determined to produce a given value of nominal lattice spacing λ.

In the N-node 1-D lattices that are considered in this report, discussion is limited to symmetric equilibrium configurations. If the notation F_j is used to denote the Cartesian external load applied to node j of the lattice (positive to the right), then the symmetric equilibrium condition is simply that $F_j = -F_{N+1-j}$.

2. The Force Approach

The force approach is typically one in which known external forces are applied to a body, which, in turn, establishes resultant internal forces in the body. If the constitutive laws governing the body are known (which in the case of a lattice means the spring force relation $f(x)$), then a deformation in the body is likewise established, associated with the applied external load. In the current analysis, however, I use the known constitutive behavior of the lattice to instead answer the question of what force must be externally applied in order to establish a given deformation.

The method for calculating the internal forces is the "free-body diagram," as described in any undergraduate book on physics[4] or statics.[5] In the case of a non-accelerating system, the body is conceptually "cut in two" and the *internal* forces across the cut of the body must be of a magnitude and direction to precisely bal-

ance the *external* forces being applied to that section of the body. In general, this force balancing requires vectorial addition; however, because the problem under consideration is a 1-D lattice, the force balance becomes a purely scalar process.

2.1 The 2-Node Structure

The archetypal 2-node interaction is shown in Fig. 1. In this interaction, the node pair $1 \backsim 2$ has been brought together to a separation distance of λ, requiring the application of symmetric forces F_1 and F_2. The function $f(x)$, here and throughout this report, is taken as the internal (compressive) spring force that acts between any 2 arbitrary nodes of the lattice, when separated by a distance x (note: x is not the displacement from a rest configuration, but rather the physical spring length!). While the convention is here employed that a positive force is compressive, tensile situations are covered with no loss in generality.
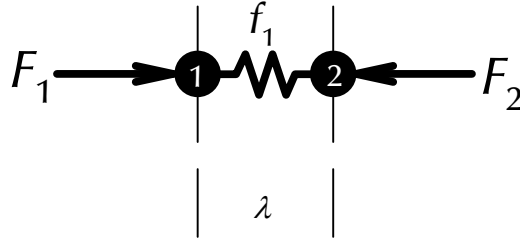
Fig. 1 A 1-D, 2-node structure with spacing λ

In this figure and throughout this report, I use the notation that $f_k \equiv f(k\lambda)$ represents the "spring" force between a pair of nodes that are separated by a distance of $s_k = k\lambda$ (in general, k need not be an integer). Thus, in Fig. 1, $f_1 \equiv f(\lambda)$. Symmetric equilibrium dictates that $F_1 = -F_2$.

In this simplest of interactions, conceptually *cutting the lattice between nodes* 1 and 2 would reveal that the internal force across the cut, call it $\hat{F}$, is comprised solely of the single spring force f_1. Enforcing a force balance over the cut "free body" (such that the external force F_1 must exactly balance the internal force $\hat{F}$) simply reveals that

$$F_1 = \hat{F} = f_1 \quad . \tag{1}$$

Thus, the separation distance x will adjust itself under the symmetrically applied load, so as to precisely generate a spring force of $f_1 = F_1$ when $x = \lambda$. The internal force $\hat{F}$ is uniform throughout the extent of the 2-node structure. Because we make

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no assumptions about the functional nature of the spring force $f(x)$, no more can be said about the relation between F and λ.

2.2 The 3-Node Lattice

The 3-node structure depicted in Fig. 2 is the simplest geometry that can be called a "lattice," because of the repetition of cell geometry. Symmetrical equilibrium is enforced as a constraint, such that F_2 (not shown in figure) is exactly zero, and $F_1 = -F_3$. Because of topologically symmetric stiffness about node 2, the uniformity of spacing between node pairs $1\text{⌇}2$ and $2\text{⌇}3$ is assured, denoted here as λ.
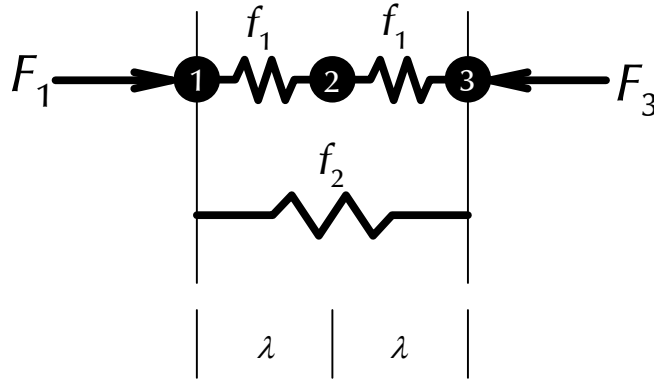
Fig. 2 A fully connected, 1-D, 3-node lattice with uniform spacing λ

Accounted for here is the effect of non-nearest-neighbor interactions, in which node pair $1\text{⌇}3$ is likewise joined by way of a "spring" interaction—the same $f(x)$ spring that governs the interactions of node pairs $1\text{⌇}2$ and $2\text{⌇}3$. However, in the case of node pair $1\text{⌇}3$, the spring force f is evaluated at a separation of 2λ, thus f_2.

Whether one conceptually cuts the structure *between* node pair $1\text{⌇}2$ or $2\text{⌇}3$, the internal force is found to be $f_1 + f_2$. Performing the balance between internal and external force leads to the result

$$F_1 = \hat{F} = f_1 + f_2 \quad . \tag{2}$$

A quick comparison to Eq. 1 reveals that the external force F_1, corresponding to a given spacing λ for the 2- and 3-node cases, is different, even when employing the same $f(x)$ spring in both cases. Such a discrepancy will manifest when spring connectivity includes non-nearest neighbors.

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2.3 Uniformly Spaced Lattice *vs.* Simply Loaded Lattice

To extend this process beyond the 3-node lattice, a choice needs to be made, for a basic incompatibility arises. On one hand, if the lattice remains simply loaded at the lattice boundary, the nodal spacing can no longer remain uniform. On the other hand, enforcing a uniform lattice spacing implies that some form of external loading must be applied even to internal nodes (one need not concern oneself with *how* that might be brought to bear).

Both possibilities are treated in the following subsections. Considered are uniformly spaced 4-, 5-, and 6-node lattices, as well as simply loaded 4- and 5-node lattices. However, some new nomenclature is required to describe the added complexity.

In the case of a uniformly spaced lattice, external forces are necessarily applied to internal nodes. With the external load being distributed over several nodes, the *net* force being applied to the lattice may be defined as

$$F_{\text{net}} = \sum_{j=1}^{N/2} F_j \quad . \tag{3}$$

Note that, when N is odd, it does not matter whether the sum index j is rounded up or down since, because of symmetric equilibrium, the central lattice node will have 0 external force being applied to it.

Because external forces are applied to internal nodes for the case of a uniformly spaced lattice, no longer is $\hat{F}$ sufficient to completely characterize the internal lattice force. Thus, $\hat{F}_j$ is used to denote the internal force in a lattice that exists between the node pair $j \curvearrowright (j+1)$ (*e.g.*, $\hat{F}_1$ is the total internal force that exists between node pair $1 \curvearrowright 2$). Note that, because of symmetric equilibrium, $\hat{F}_j = \hat{F}_{N-j}$.

The nomenclature s_k has already been introduced as shorthand for the distance $k\lambda$. In a related way, the syntax of $_i^m x$ is provided to mean the physical separation between node pair $i \curvearrowright m$ of the lattice. For the special case of a uniformly spaced lattice, $_i^m x \equiv s_{|m-i|} \equiv |m-i|\lambda$. Whereas the notation $_i^m x$ provides the specificity of both nodes in the node pair, the notation s_k allows the convenience of tracking only the separation distance in node pair $i \curvearrowright m$, without the need to know the specific values of i and m.

The presence of a spatially dependent internal force allows for the consideration of

an *average* internal force within the lattice. In the most general consideration,

$$\bar{F} = \frac{1}{L} \int_0^L \hat{F}(x)\,dx$$

$$= \sum_{j=1}^{N-1} \left(\hat{F}_j \cdot {}^{j+1}_{\ j}x \right) \bigg/ \sum_{j=1}^{N-1} {}^{j+1}_{\ j}x \tag{4}$$

However, when the lattice is uniformly spaced, the above form is simplified, by knowing that ${}^{j+1}_{\ j}x = \lambda$:

$$\bar{F} = \frac{1}{N-1} \sum_{j=1}^{N-1} \hat{F}_j \quad, \text{ (uniform lattice spacing).} \tag{5}$$

In the case of a simply loaded lattice, however, the spacing within the lattice will not remain uniform. Thus, we introduce the nomenclature of Δ as a fraction of λ representing the deviation in lattice spacing from the nominal value. In this way, $f_{1-\Delta}$ represents the spring force evaluated at a nodal separation of $\lambda(1-\Delta)$. With a suitably large lattice, one would require indexed Δ_j to represent different deviations from the nominal lattice spacing for different node pairs. However, for the scope of this report, a single, unindexed Δ is sufficient.

2.4 The Uniformly Spaced 4-Node Lattice

In Fig. 3 is found a 4-node lattice in which uniform spacing is enforced. In order to maintain uniform spacing, there must be, in addition to the forces F_1 and F_4 applied at the external boundaries, forces F_2 and F_3 applied upon the internal nodes.

Conceptually cutting the lattice *between* each successive node pair, the internal forces in the lattice are revealed as

$$\hat{F}_1 = f_1 + \ f_2 + f_3 \quad, \tag{6}$$

$$\hat{F}_2 = f_1 + 2f_2 + f_3 \quad, \tag{7}$$

and

$$\hat{F}_3 = \hat{F}_1 \quad. \tag{8}$$
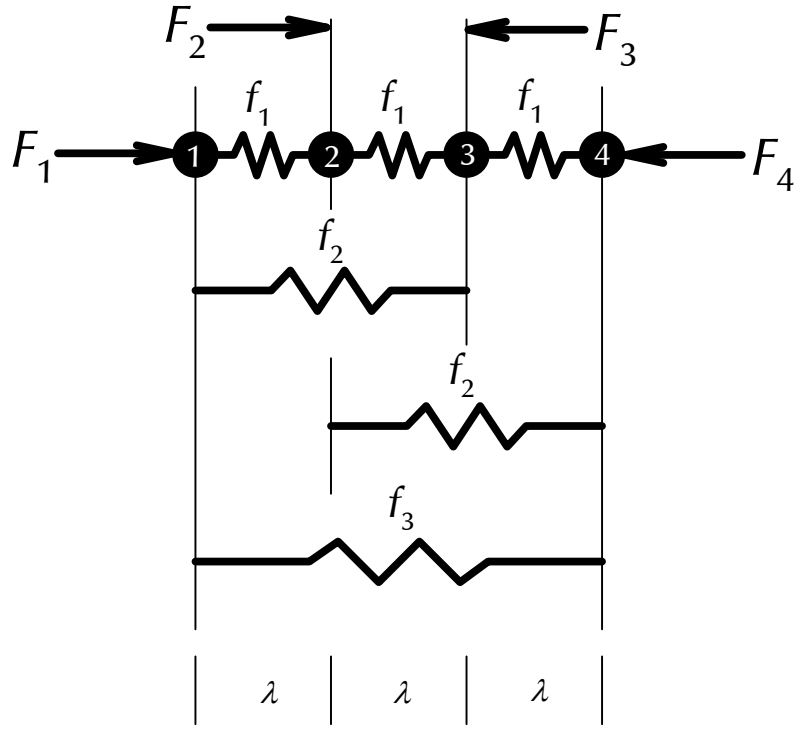
Fig. 3 A fully connected, 1-D, 4-node lattice with uniform spacing λ

With this knowledge, the average internal lattice force may be determined as

$$\begin{aligned}
\bar{F} &= \frac{1}{3}\left(\hat{F}_1 + \hat{F}_2 + \hat{F}_3\right) \\
&= f_1 + \frac{4}{3}f_2 + f_3
\end{aligned} \tag{9}$$

If one performs the internal/external force balance with each instance of lattice "cut," the free-body diagrams reveal that

$$\hat{F}_1 = F_1 \quad , \qquad \hat{F}_2 = F_1 + F_2 \quad . \tag{10}$$

Solving for the external forces reveals

$$F_1 = \hat{F}_1 \qquad = f_1 + f_2 + f_3 \quad , \tag{11}$$

$$F_2 = \hat{F}_2 - \hat{F}_1 = \qquad f_2 \tag{12}$$

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with $F_4 = -F_1$ and $F_3 = -F_2$. The net externally applied load is

$$\begin{aligned} F_{\text{net}} &= F_1 + F_2 = \hat{F}_2 \\ &= f_1 + 2f_2 + f_3 \end{aligned}$$

(13)

Because the net external load is not uniformly present across the lattice, it is seen to differ from the average internal load, $\bar{F}$, given in Eq. 9.

As a purely illustrative example, consider a spring with the relation $f(x) \propto 1/x^2$. For such a hypothetical example, the significance of F_2 compared to the net load F_{net} can be evaluated from Eqs. 12 and 13 as

$$\frac{F_2}{F_{\text{net}}} = \frac{1/2^2}{1/1^2 + 2/2^2 + 1/3^2} = \frac{9}{58} \approx 0.156 \quad .$$

(14)

For this hypothetical example, the significance of F_2 relative to F_{net} is independent of λ. That is by no means guaranteed, when considering cases involving more complex functional relationships for $f(x)$.

2.5 The Uniformly Spaced 5-Node Lattice

Consider the uniformly spaced 5-node lattice in Fig. 4. It may be useful to note that the total number of springs in a fully connected N-node lattice is $N(N-1)/2$; namely, $N-1$ springs with a separation of s_1, $N-2$ springs with a separation of s_2, ... , and 1 spring with a separation of $s_{(N-1)}$.

As before, conceptual cuts are made through the lattice *between each successive node pair*, in order to reveal the internal loads:

$$\hat{F}_1 = f_1 + \ f_2 + \ f_3 + f_4 \quad , $$

(15)

$$\hat{F}_2 = f_1 + 2f_2 + 2f_3 + f_4 \quad , $$

(16)

and

$$\hat{F}_3 = \hat{F}_2 \quad , \qquad \hat{F}_4 = \hat{F}_1 \quad . $$

(17)

From these, the average internal force may be calculated as

$$\begin{aligned} \bar{F} &= \frac{1}{4}\left(\hat{F}_1 + \hat{F}_2 + \hat{F}_3 + \hat{F}_4 \right) \\ &= f_1 + \frac{3}{2}f_2 + \frac{3}{2}f_3 + f_4 \end{aligned}$$

(18)
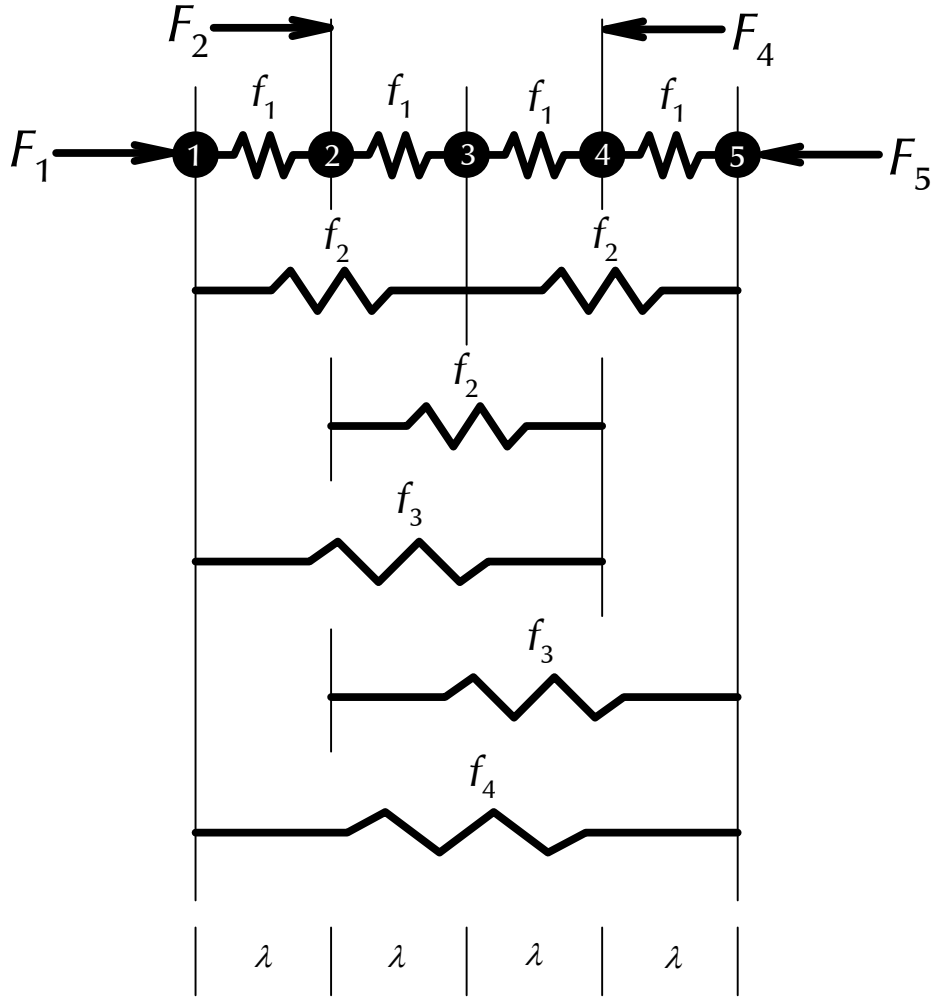
Fig. 4 A fully connected, 1-D, 5-node lattice with uniform spacing λ

A force balance at each instance of lattice cut reveals the internal forces:

$$\hat{F}_1 = F_1 \quad , \qquad \hat{F}_2 = F_1 + F_2 \quad . \tag{19}$$

These 2 equations may be solved for the external forces:

$$F_1 = \hat{F}_1 \qquad = f_1 + f_2 + f_3 + f_4 \quad , \tag{20}$$

$$F_2 = \hat{F}_2 - \hat{F}_1 = \qquad f_2 + f_3 \quad , \tag{21}$$

with $F_5 = -F_1$, $F_4 = -F_2$ and $F_3 = 0$, by way of symmetric equilibrium. The net

externally applied load is, therefore,

$$F_{\text{net}} = F_1 + F_2 = \hat{F}_2$$
$$= f_1 + 2f_2 + 2f_3 + f_4 \qquad (22)$$

The net applied lattice force, F_{net}, is seen to differ from the average internal force, $\bar{F}$, in the lattice, given by Eq. 18.

As before, we can compute the significance of F_2 relative to F_{net}, employing a hypothetical spring for which $f(x) \propto 1/x^2$. The result is

$$\frac{F_2}{F_{\text{net}}} = \frac{1/2^2 + 1/3^2}{1/1^2 + 2/2^2 + 2/3^2 + 1/4^2} = \frac{52}{257} \approx 0.202 \quad . \qquad (23)$$

2.6 The Uniformly Spaced 6-Node Lattice

The final case of a uniformly spaced lattice considered in this report is that comprised of 6 nodes, as depicted in Fig. 5. Making conceptual cuts through the lattice *between* each successive node pair is done to evaluate the internal loads:

$$\hat{F}_1 = f_1 + f_2 + f_3 + f_4 + f_5 \qquad (24)$$
$$\hat{F}_2 = f_1 + 2f_2 + 2f_3 + 2f_4 + f_5 \qquad (25)$$
$$\hat{F}_3 = f_1 + 2f_2 + 3f_3 + 2f_4 + f_5 \qquad (26)$$

and

$$\hat{F}_4 = \hat{F}_2 \quad , \qquad \hat{F}_5 = \hat{F}_1 \quad . \qquad (27)$$

As in prior configurations, the average internal force may be evaluated as

$$\bar{F} = \frac{1}{5}\left(\hat{F}_1 + \hat{F}_2 + \hat{F}_3 + \hat{F}_4 + \hat{F}_5\right)$$
$$= f_1 + \frac{8}{5}f_2 + \frac{9}{5}f_3 + \frac{8}{5}f_4 + f_5 \qquad (28)$$

A force balance, performed at each conceptual lattice cut, reveals the magnitude of the internal forces:

$$\hat{F}_1 = F_1 \quad , \qquad \hat{F}_2 = F_1 + F_2; \quad , \qquad \hat{F}_3 = F_1 + F_2 + F_3 \quad . \qquad (29)$$
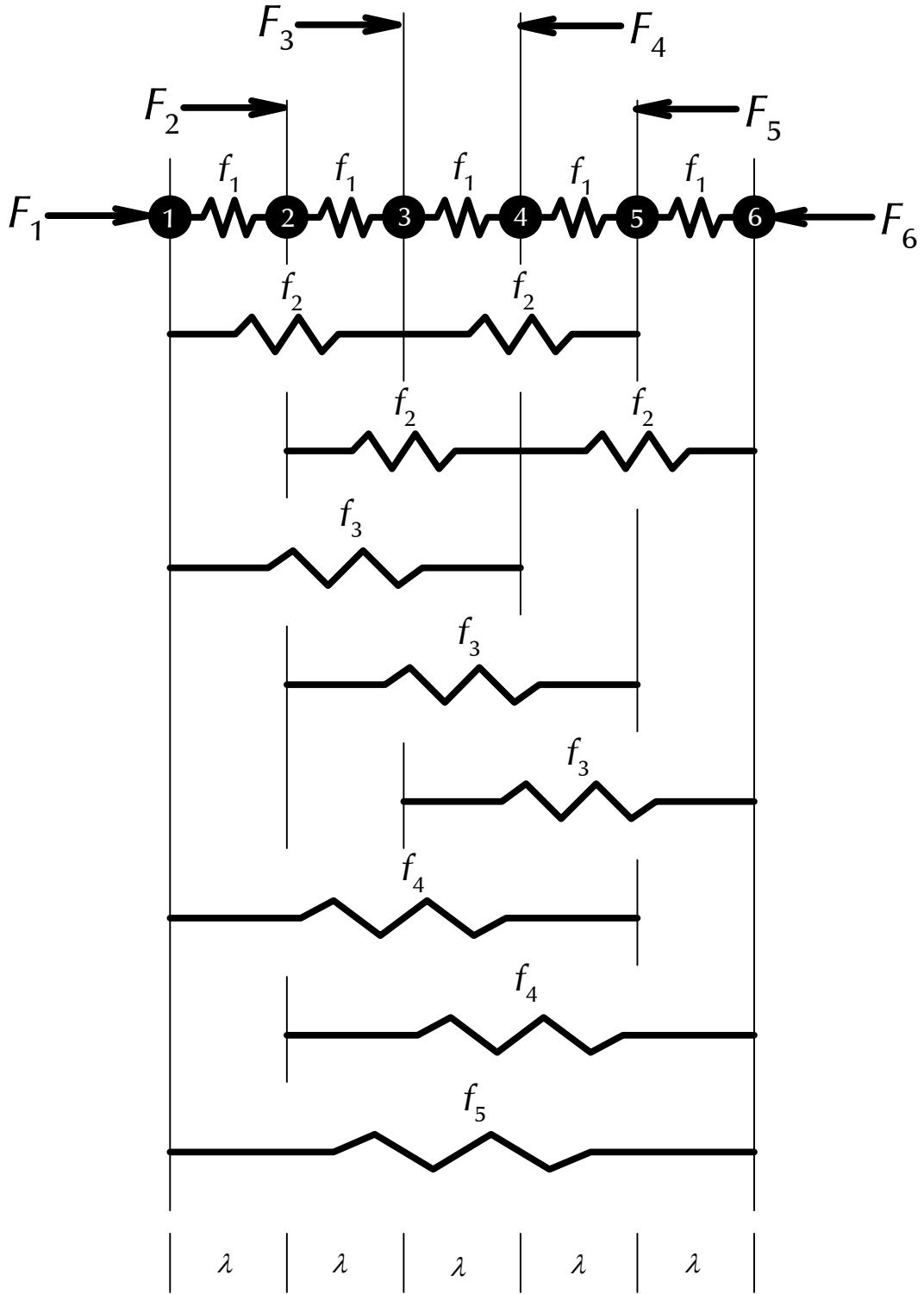
Fig. 5 A fully connected, 1-D, 6-node lattice with uniform spacing λ

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These 3 equations may be solved for the applied external forces:

$$F_1 = \hat{F}_1 \qquad = f_1 + f_2 + f_3 + f_4 + f_5 \quad , \tag{30}$$

$$F_2 = \hat{F}_2 - \hat{F}_1 = \qquad f_2 + f_3 + f_4 \quad , \tag{31}$$

$$F_3 = \hat{F}_3 - \hat{F}_2 = \qquad\qquad f_3 \quad , \tag{32}$$

with $F_6 = -F_1$, $F_5 = -F_2$, and $F_4 = -F_3$. The net external load (again distinct from $\bar{F}$) is therefore

$$\begin{aligned} F_{\text{net}} &= F_1 + F_2 + F_3 = \hat{F}_3 \\ &= f_1 + 2f_2 + 3f_3 + 2f_4 + f_5 \end{aligned} \tag{33}$$

For a hypothetical lattice spring in which $f(x) \propto 1/x^2$, the relative significance of both F_2 and F_3 may be evaluated:

$$\frac{F_2}{F_{\text{net}}} = \frac{1/2^2 + 1/3^2 + 1/4^2}{1/1^2 + 2/2^2 + 3/3^2 + 2/4^2 + 1/5^2} = \frac{1525}{7194} \approx 0.212 \tag{34}$$

and

$$\frac{F_3}{F_{\text{net}}} = \frac{1/3^2}{1/1^2 + 2/2^2 + 3/3^2 + 2/4^2 + 1/5^2} = \frac{400}{7194} \approx 0.056 \quad . \tag{35}$$

Of course, a different functional relationship for the lattice spring would imply different quantitative values for the significance of F_2 and F_3.

2.7 The Simply Loaded 4-Node Lattice

Consider now a lattice that is simply loaded at its boundaries. While the internal load $\hat{F}_j$ may be evaluated by conceptually cutting the lattice between any 2 adjacent nodes j and $j + 1$, the absence of external forces at the internal nodes implies that the internal load is everywhere uniform, or $\hat{F}_j \equiv \hat{F}$. Figure 6 depicts such a lattice, comprising 4 nodes. Because the internodal force relationships differ for the internal nodes $(2, 3)$ *vis-à-vis* the boundary nodes $(1, 4)$, the interlattice spacing of node pairs $1 \leadsto 2$ and $3 \leadsto 4$ will differ by an amount $\Delta\lambda$ from that of node pair $2 \leadsto 3$. The lattice spacing of node pair $2 \leadsto 3$ is designated λ, since that pair is the farthest from the influence of boundary effects.
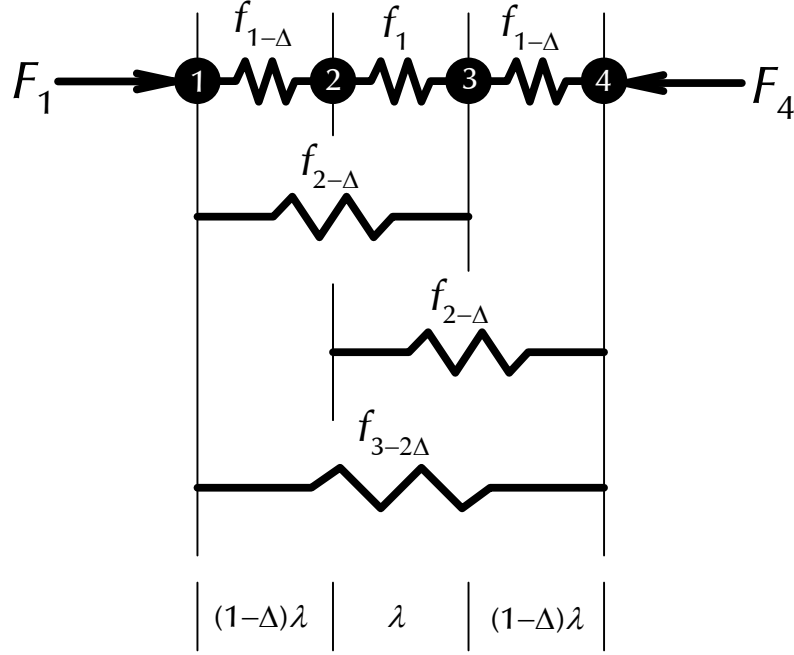
Fig. 6 A fully connected, 1-D, 4-node lattice with nominal spacing λ and simple boundary loading

Performing a conceptual cut *between* each successive node pair so as to tabulate the internal forces gives

$$\hat{F}_1 = f_{1-\Delta} + \ f_{2-\Delta} + f_{3-2\Delta} \quad , \tag{36}$$

$$\hat{F}_2 = f_1 \quad + 2f_{2-\Delta} + f_{3-2\Delta} \quad , \tag{37}$$

with $\hat{F}_3 = \hat{F}_1$. However, in the case of a simply loaded lattice, the internal loads must be everywhere uniform. Thus, $\hat{F} \equiv \hat{F}_1 \equiv \hat{F}_2$, leading to the constraint on Δ that

$$f_1 + f_{2-\Delta} = f_{1-\Delta} \quad . \tag{38}$$

The external load F_1, being equal to $\hat{F}$, can therefore be expressed in several ways:

$$F_1 = \hat{F} = \begin{cases} f_{1-\Delta} + \ f_{2-\Delta} + f_{3-2\Delta} \\ f_1 \quad + 2f_{2-\Delta} + f_{3-2\Delta} \end{cases} \quad . \tag{39}$$

Only if the spring function is explicitly defined may the value of Δ be quantitatively ascertained. If, for example, the lattice spring obeyed $f(x) \propto 1/x^2$, then the

constraint, Eq. 38, would take the form

$$\frac{1}{1^2} + \frac{1}{(2-\Delta)^2} = \frac{1}{(1-\Delta)^2} \quad , \tag{40}$$

which may be numerically solved for $\Delta \approx 0.117$ (limiting Δ to real values in the range $\Delta < 1$). Thus, an 11.7% variation in lattice spacing would be evident across a simply loaded 4-node lattice, for which $f(x) \propto 1/x^2$ governed the spring force.

2.8 The Simply Loaded 5-Node Lattice

The final lattice that is considered in this report is a simply loaded 5-node lattice (Fig. 7).
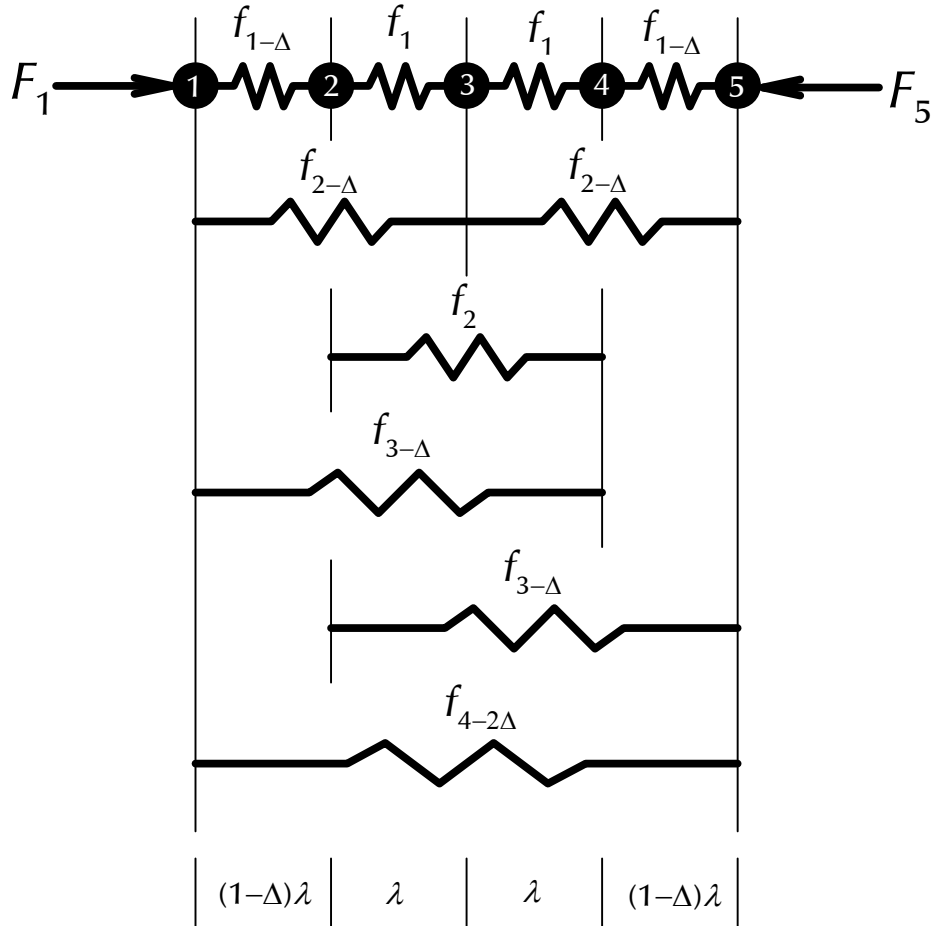
Fig. 7 A fully connected, 1-D, 5-node lattice with nominal spacing λ and simple boundary loading

The force balance may proceed at each conceptual cut across the lattice, *between*

successive nodes:

$$\hat{F}_1 = f_{1-\Delta} \quad + f_{2-\Delta} + \ f_{3-\Delta} + f_{4-2\Delta} \quad , \tag{41}$$

$$\hat{F}_2 = f_1 + f_2 + f_{2-\Delta} + 2f_{3-\Delta} + f_{4-2\Delta} \quad , \tag{42}$$

and

$$\hat{F}_3 = \hat{F}_2 \quad , \qquad \hat{F}_4 = \hat{F}_1 \quad . \tag{43}$$

Since a uniform internal load is manifested throughout the lattice, it is the case that $\hat{F} \equiv \hat{F}_1 \equiv \hat{F}_2$, which leads to the constraint on Δ that

$$f_{1-\Delta} = f_1 + f_2 + f_{3-\Delta} \quad . \tag{44}$$

The external load F_1, being equal to $\hat{F}$, can be expressed in several ways:

$$F_1 = \hat{F} = \begin{cases} f_{1-\Delta} \quad + f_{2-\Delta} + \ f_{3-\Delta} + f_{4-2\Delta} \\ f_1 + f_2 \ + f_{2-\Delta} + 2f_{3-\Delta} + f_{4-2\Delta} \end{cases} . \tag{45}$$

If a spring function is specified, the value of Δ will be determinable. Considering a hypothetical spring for which $f(x) \propto 1/x^2$, the constraint, Eq. 44, becomes

$$\frac{1}{(1-\Delta)^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{(3-\Delta)^2} \quad , \tag{46}$$

which can be numerically evaluated to ascertain that $\Delta \approx 0.147$ (limited to real solutions for which $\Delta < 1$).

3. The Energy Approach

In Section 2, a force approach was used to develop the equations of equilibrium in a series of 1-D lattices. When nodes in the lattice are not accelerating, equilibrium is established, with the help of free-body diagrams, by developing a balance between internal and external forces. A general formulaic tabulation of equilibrium for a 1-D lattice of arbitrary size was not sought using the force approach. Rather, individual 1-D lattices, ranging from 2 to 6 nodes, were examined, under 2 separate behavioral assumptions: that of constant lattice spacing and that of simple loading at the lattice boundary. The process was a tedious one, in which a full accounting of internodal interactions was ascertained and tabulated. Were the lattices in 2 or 3 dimensions rather than 1, the formulations would be still more intricate, as vectorial calculations

would be required for component resolution.

In the force approach, the concept of energy was totally absent from the analysis. In this section, the goal is to re-examine those same lattices using an energy approach, in which force is a quantity derived from the energy. The goal is to establish a compatible result.

In a lattice, let the potential energy that any 2 nodes possess as a result of their mutual interaction at a separation distance of x be given as $e(x)$. If the nodes are materially indistinguishable, then half of that potential energy may be partitioned to each of the 2 interacting nodes. The energy associated with node i of an N-node lattice is, therefore, the sum of its interactions with all the other nodes of the lattice:

$$E_i = \frac{1}{2} \sum_{m=1}^{N} e\binom{m}{i}x) \quad , \tag{47}$$

where $e(0)$ is taken as 0 to address the interaction of node i with itself, and where the factor of $1/2$ accounts for the partition of potential energy to both nodes participating in an interaction (*i.e.*, only $1/2$ the interaction energy belongs to node i). The total energy of the lattice is obtained by repeating this process over all the nodes of the lattice.

$$E = \frac{1}{2} \sum_{i=1}^{N} \sum_{m=1}^{N} e\binom{m}{i}x) \quad . \tag{48}$$

The macroscopic force associated with changing the dimension of the overall lattice structure is derived from the total energy as

$$F = -\frac{dE}{dL} \quad , \tag{49}$$

whereas the force of single internodal interaction at a spacing of s is, correspondingly,

$$f = -\frac{de}{ds} \quad . \tag{50}$$

The key question examined in this report is determining whether these derived forces (Eqs. 49 and 50) in the context of Eq. 48 are compatible with the earlier derived force approach of Section 2.

A prerequisite in addressing this question is understanding the relationship between a change in the global lattice size (L) and the corresponding change in the lattice

spacing of node pair $i \wedge m$, which is given by $_i^m x$. To do so, the chain rule is applied,

$$\frac{de\left(_i^m x\right)}{dL} = \frac{de\left(_i^m x\right)}{d_i^m x} \cdot \frac{d_i^m x}{d\lambda} \cdot \frac{d\lambda}{dL} \quad . \tag{51}$$

The first term of the right-hand side is simply

$$\frac{de\left(_i^m x\right)}{d_i^m x} = -f\left(_i^m x\right) \quad , \tag{52}$$

by definition. To the first order, the last term of Eq. 51, under simple expansion, is

$$\frac{d\lambda}{dL} = \frac{\lambda}{L} \quad , \tag{53}$$

which is independent of the node pair $i \wedge m$. Equation 51 may thus be employed when taking the derivative of the energy equation, Eq. 48, with respect to the body dimension L, to obtain the macroscopic force:

$$F = \frac{\lambda}{2L} \sum_{i=1}^{N} \sum_{m=1}^{N} f\left(_i^m x\right) \cdot \frac{d_i^m x}{d\lambda} \quad . \tag{54}$$

Similar to energy, $f(0) \equiv 0$ to remove any effect of self force. The multiplication of the local force by $d_i^m x / d\lambda$ is key to a proper force enumeration and has been seen before in my previous work on lattice force[6–8] (as $ds/d\lambda$).

3.1 Nodal Forces

Whereas Section 3 provides the means to evaluate the global lattice force in terms of the spring-component forces, by way of an energy derivative, it does not provide the means to evaluate the component nodal forces. To obtain the externally applied nodal forces from the energy approach, one has to imagine a connected lattice in which 1 of the nodes, i, is displaced from its equilibrium position by an infinitesimal amount, $\lambda \, d\epsilon_i$. The pre-existing applied nodal force that moves through that displacement causes an energy change such that

$$F_i = \frac{dE}{\lambda \, d\epsilon_i} \quad . \tag{55}$$

Since E is composed of a sum of internodal e terms, one can evaluate the derivative using a chain rule:

$$\frac{de\left(_i^m x\right)}{\lambda \, d\epsilon_i} = \frac{de\left(_i^m x\right)}{d_i^m x} \cdot \frac{d_i^m x}{\lambda \, d\epsilon_i} \quad . \tag{56}$$

Therefore,

$$F_i = -\frac{1}{2} \sum_{\bar{\imath}=1}^{N} \sum_{m=1}^{N} f(\,{}_{\bar{\imath}}^{m}x) \cdot \frac{d\,{}_{\bar{\imath}}^{m}x}{\lambda \, d\epsilon_i} \quad . \tag{57}$$

When neither of the indices, $\bar{\imath}$ nor m, correspond to the displaced node i, the distance ${}_{\bar{\imath}}^{m}x$ remains unchanged and thus, $d\,{}_{\bar{\imath}}^{m}x/d\epsilon_i = 0$ $(i \neq \bar{\imath}, m)$. Also, the distance ${}_{\bar{\imath}}^{m}x$ does not depend on the order of i and m, so that ${}_{i}^{m}x \equiv {}_{m}^{i}x$. This symmetry, and the requirement that either $\bar{\imath}$ or m take on the value of i for a non-zero term to arise, allows for a condensation of Eq. 57:

$$F_i = -\sum_{m=1}^{N} f(\,{}_{i}^{m}x) \cdot \frac{d\,{}_{i}^{m}x}{\lambda \, d\epsilon_i} \quad . \tag{58}$$

Note that, because ϵ is infinitesimal, one may safely approximate $f_{k \pm \epsilon}$ as f_k in the evaluation of $f(\,{}_{i}^{m}x)$ terms in Eq. 58. However, given an infinitesimal positive displacement of node i, the final term of Eq. 58 becomes

$$\frac{d\,{}_{i}^{m}x}{\lambda \, d\epsilon_i} = \begin{cases} 1 & (m < i) \\ 0 & (m = i) \\ -1 & (m > i) \end{cases} . \tag{59}$$

When a lattice is uniformly spaced, the resultant F_i indicates the external force being applied to node i in order to maintain the uniform spacing.

For the case of the simply loaded lattice, the boundary load may be determined by choosing $i = 1$ (or $i = N$) for the evaluation of Eq. 58. However, when i is chosen as an internal node of a simply loaded lattice, it is known that the applied force on internal nodes is zero. Therefore, setting the resultant force equivalent to 0 instead generates a constraint equation on the internodal forces, of the kind indicated in Eqs. 38 and 44. These constraints can be used, once a lattice-spring force relation is specified, to evaluate the nonuniformity (Δ) of the equilibrium lattice spacing for a simply loaded lattice.

We now proceed to reanalyze the lattices of Section 2, in light of Eqs. 54 and 58. For the analysis, the shorthand $e_k \equiv e(s_k) \equiv e(k\lambda)$ is introduced, in a fashion analogous to f_k.

3.2 Simplification for Uniformly Spaced Lattice Structures

For a uniformly spaced lattice, in which ${}^m_i x = s_{|m-i|} = |m - i|\lambda$, the second and last terms in the derivative chain of Eq. 51 immediately follows as

$$\frac{d^m_i x}{d\lambda} = \frac{ds_{|m-i|}}{d\lambda} = |m - i| \quad , \text{ (uniform lattice spacing)} \tag{60}$$

and

$$\frac{d\lambda}{dL} = \frac{\lambda}{L} = \frac{1}{N - 1} \quad , \text{ (uniform lattice spacing).} \tag{61}$$

On the other hand, when the lattice is simply loaded, Eq. 60 needs to be modified to account for the Δ terms associated with the nonuniformity of lattice spacing for the node pair $i \curlywedge m$. Nonetheless, under simple expansion, to the first order

$$\frac{d^m_i x}{d\lambda} = \frac{{}^m_i x}{\lambda} \quad . \tag{62}$$

Likewise, Eq. 61 will necessarily revert to Eq. 53 for a simply loaded lattice— nonetheless, λ/L remains a parameter independent of i and m, accounting for its removal from the double summation of Eq. 54.

3.3 The 2-Node Structure

For this discussion, please refer to Fig. 1. The simplifications of Section 3.2 apply to this case. As the lattice size (N) increases (in later report sections), it helps organize the mind first by tabulating the nodal energies (Eq. 47):

$$2E_1 = e_1$$
$$2E_2 = e_1$$
$$E = \sum_{i=1}^{2} E_i = e_1 \tag{63}$$

A direct application of Eq. 54 gives

$$F = \frac{1}{1}\left[f_1 \cdot 1\right] ,$$
$$= f_1 \tag{64}$$

which matches Eq. 1 as expected.

3.4 The 3-Node Lattice

For this discussion, please refer to Fig. 2. The simplifications of Section 3.2 apply to this case. The nodal energy tabulation (Eq. 47) is

$$
\begin{aligned}
2E_1 &= e_1 + e_2 \\
2E_2 &= 2e_1 \\
2E_3 &= e_1 + e_2 \\
E &= \sum_{i=1}^{3} E_i = 2e_1 + e_2
\end{aligned}
\tag{65}
$$

Recall, the subscript on E_i refers to a node number, whereas the subscript on e_k refers to internodal separation as a multiple of λ. A direct application of Eq. 54 gives

$$
\begin{aligned}
F &= \frac{1}{2}\left[2f_1 \cdot 1 + f_2 \cdot 2\right] \\
&= f_1 + f_2
\end{aligned}
\tag{66}
$$

which matches Eq. 2, as derived using a force approach.

3.5 The Uniformly Spaced 4-Node Lattice

For this discussion, please refer to Fig. 3. The simplifications of Section 3.2 apply to this case. A tabulation of the nodal energies (Eq. 47) gives

$$
\begin{aligned}
2E_1 &= e_1 + e_2 + e_3 \\
2E_2 &= 2e_1 + e_2 \\
2E_3 &= 2e_1 + e_2 \\
2E_4 &= e_1 + e_2 + e_3 \\
E &= \sum_{i=1}^{4} E_i = 3e_1 + 2e_2 + e_3
\end{aligned}
\tag{67}
$$

A direct application of Eq. 54 gives

$$
\begin{aligned}
F &= \frac{1}{3}\left[3f_1 \cdot 1 + 2f_2 \cdot 2 + f_3 \cdot 3\right] \\
&= f_1 + \frac{4}{3}f_2 + f_3
\end{aligned}
\tag{68}
$$

A comparison to the force approach of Section 2.4 gives an interesting revelation.

Because a uniformly spaced lattice has multiple points of external force application, there is not a simple value of F against which to compare Eq. 68. Instead, there is the net applied force (F_{net}) of Eq. 13 and the average internal force ($\bar{F}$) of Eq. 9. Interestingly, this value of macroscopic force F given by Eq. 68 matches $\bar{F}$, the average internal force inside the lattice.

To generate the applied nodal forces, the relation Eq. 58 needs to be applied to successive nodes of the uniformly spaced 4-node lattice. A direct application to node 1 gives

$$F_1 = -\left[f_1 \cdot (-1) + f_2 \cdot (-1) + f_3 \cdot (-1) \right] \quad , \tag{69}$$

which simplifies to

$$F_1 = f_1 + f_2 + f_3 \quad , \tag{70}$$

which matches the expression given in Eq. 11 for the node 1 loading of the uniformly spaced 4-node lattice. A direct application to node 2 gives

$$F_2 = -\left[f_1 \cdot (1) + f_1 \cdot (-1) + f_2 \cdot (-1) \right]$$
$$= f_2 \quad , \tag{71}$$

which matches the expression given in Eq. 12 for the node 2 loading of the uniformly spaced 4-node lattice.

3.6 The Uniformly Spaced 5-Node Lattice

For this discussion, please refer to Fig. 4. The simplifications of Section 3.2 apply to this case. A tabulation of the nodal energies (Eq. 47) gives

$$2E_1 = e_1 + e_2 + e_3 + e_4$$
$$2E_2 = 2e_1 + e_2 + e_3$$
$$2E_3 = 2e_1 + 2e_2$$
$$2E_4 = 2e_1 + e_2 + e_3$$
$$2E_5 = e_1 + e_2 + e_3 + e_4$$
$$E = \sum_{i=1}^{5} E_i = 4e_1 + 3e_2 + 2e_3 + e_4 \tag{72}$$

The pattern in the e_k term multipliers can be inferred as the number of springs present at a distance of 1λ, 2λ, 3λ, *etc.*, respectively, in the given lattice configura-

tion. A direct application of Eq. 54 gives

$$F = \frac{1}{4}\left[4f_1 \cdot 1 + 3f_2 \cdot 2 + 2f_3 \cdot 3 + f_4 \cdot 4\right]$$
$$= f_1 + \frac{3}{2}f_2 + \frac{3}{2}f_3 + f_4 \qquad\qquad (73)$$

As in the case of the 4-node lattice, the value of macroscopic force F given by Eq. 73 matches $\bar{F}$, the average internal force inside the 5-node lattice, as given by Eq. 18.

To generate the applied nodal forces, the relation Eq. 58 needs to be applied to successive nodes of the uniformly spaced 5-node lattice. A direct application to nodes 1–3, respectively, gives

$$F_1 = -\left[f_1 \cdot (-1) + f_2 \cdot (-1) + f_3 \cdot (-1) + f_4 \cdot (-1)\right]$$
$$= f_1 + f_2 + f_3 + f_4 \qquad\qquad (74)$$
$$F_2 = -\left[f_1 \cdot (1) + f_1 \cdot (-1) + f_2 \cdot (-1) + f_3 \cdot (-1)\right]$$
$$= f_2 + f_3 \qquad\qquad (75)$$
$$F_3 = -\left[f_2 \cdot (1) + f_1 \cdot (1) + f_1 \cdot (-1) + f_2 \cdot (-1)\right]$$
$$= 0 \qquad\qquad (76)$$

matching the expressions given in Eqs. 20–21 for the loading of the uniformly spaced 5-node lattice. An application to node 3 gives the expected result that $F_3 = 0$.

3.7 The Uniformly Spaced 6-Node Lattice

For this discussion, please refer to Fig. 5. The simplifications of Section 3.2 apply

to this case. A tabulation of the nodal energies (Eq. 47) gives

$$2E_1 = \ e_1 + \ e_2 + e_3 + e_4 + e_5$$
$$2E_2 = 2e_1 + \ e_2 + e_3 + e_4$$
$$2E_3 = 2e_1 + 2e_2 + e_3$$
$$2E_4 = 2e_1 + 2e_2 + e_3$$
$$2E_5 = 2e_1 + \ e_2 + e_3 + e_4$$
$$2E_6 = \ e_1 + \ e_2 + e_3 + e_4 + e_5$$

$$E = \sum_{i=1}^{6} E_i = 5e_1 + 4e_2 + 3e_3 + 2e_4 + e_5 \tag{77}$$

A direct application of Eq. 54 gives

$$F = \frac{1}{5}\left[5f_1 \cdot 1 + 4f_2 \cdot 2 + 3f_3 \cdot 3 + 2f_4 \cdot 4 + f_5 \cdot 5\right]$$
$$= f_1 + \frac{8}{5}f_2 + \frac{9}{5}f_3 + \frac{8}{5}f_4 + f_5 \tag{78}$$

As in the case of the 5-node lattice, the value of macroscopic force F given by Eq. 78 matches $\bar{F}$, the average internal force inside the 5-node lattice, as given by Eq. 28.

To generate the applied nodal forces, the relation Eq. 58 needs to be applied to successive nodes of the uniformly spaced 6-node lattice. A direct application to nodes 1–3, respectively, gives

$$F_1 = -\left[f_1 \cdot (-1) + f_2 \cdot (-1) + f_3 \cdot (-1) + f_4 \cdot (-1) + f_5 \cdot (-1)\right]$$
$$= f_1 + f_2 + f_3 + f_4 + f_5 \tag{79}$$
$$F_2 = -\left[f_1 \cdot (1) + f_1 \cdot (-1) + f_2 \cdot (-1) + f_3 \cdot (-1) + f_4 \cdot (-1)\right]$$
$$= f_2 + f_3 + f_4 \tag{80}$$
$$F_3 = -\left[f_2 \cdot (1) + f_1 \cdot (1) + f_1 \cdot (-1) + f_2 \cdot (-1) + f_3 \cdot (-1)\right]$$
$$= f_3 \tag{81}$$

matching the expressions given in Eqs. 30–32 for the loading of the uniformly spaced 6-node lattice.

3.8   Summary of Uniformly Spaced N-Node Lattice Equilibrium

The solution of the uniformly spaced, fully connected 1-D-lattice equilibrium equations has been shown using both force and energy approaches. Compatibility in the 2 approaches has been established, whereby it was shown that the force derived from the derivative of energy with respect to a displacement corresponds to the average internal force in the uniformly spaced lattice (and not to the net externally applied load across the various load points).

The regularity established in both the energy and force equations allows for the generalized pattern to be recognized that connects the internodal quantities (e_k and f_k) to the macroscopic global properties (E and F):

$$E = \sum_{i=1}^{N-1} (N - i) e_i \tag{82}$$

$$F = \frac{1}{N-1} \sum_{i=1}^{N-1} i(N - i) f_i \quad . \tag{83}$$

The applied nodal forces and their net sum, for a uniformly spaced lattice, can be reduced to the following summations:

$$F_i = \sum_{m=i}^{N-i} f_m \quad , \tag{84}$$

$$F_{\text{net}} = \sum_{i=1}^{N/2} \sum_{m=i}^{N-i} f_m \quad . \tag{85}$$

A tabulated comparison of F to F_{net} is given in the table, out to $N = 8$.

Table A comparison of global *vs.* net-applied lattice force for uniformly spaced lattices

N	$F = \hat{F}$	F_{net}	$F - F_{\text{net}}$
2	f_1	f_1	0
3	$f_1 + f_2$	$f_1 + f_2$	0
4	$f_1 + \frac{4}{3}f_2 + f_3$	$f_1 + 2f_2 + f_3$	$\frac{2}{3}f_2$
5	$f_1 + \frac{3}{2}f_2 + \frac{3}{2}f_3 + f_4$	$f_1 + 2f_2 + 2f_3 + f_4$	$\frac{2}{4}f_2 + \mathcal{O}(f_3)$
6	$f_1 + \frac{8}{5}f_2 + \frac{9}{5}f_3 + \frac{8}{5}f_4 + f_5$	$f_1 + 2f_2 + 3f_3 + 2f_4 + f_5$	$\frac{2}{5}f_2 + \mathcal{O}(f_3)$
7	$f_1 + \frac{5}{3}f_2 + 2f_3 + 2f_4 + \frac{5}{3}f_5 + f_6$	$f_1 + 2f_2 + 3f_3 + 3f_4 + 2f_5 + f_6$	$\frac{2}{6}f_2 + \mathcal{O}(f_3)$
8	$f_1 + \frac{12}{7}f_2 + \frac{15}{7}f_3 + \frac{16}{7}f_4 + \frac{15}{7}f_5 + \frac{12}{7}f_6 + f_7$	$f_1 + 2f_2 + 3f_3 + 4f_4 + 3f_5 + 2f_6 + f_7$	$\frac{2}{7}f_2 + \mathcal{O}(f_3)$

An examination of the last column of the table shows the discrepancy between the global (F) and net-applied (F_{net}) lattice force. As the lattice size N grows, it can be seen that the discrepancy is $\frac{2}{N}f_2$ plus higher order terms. Considering the case where the lattice-spring stiffness is monotonically decreasing beyond a certain finite level of internodal separation (as in the case of atomic lattices), one may conclude, for large lattices (where $N \gg 2$), that F and F_{net} converge to the same value.

The simply loaded lattices for $N = 4$ and $N = 5$ are now examined. However, the simplifying relations of Section 3.2 may not be employed, because of the nonuniform spacing of the lattice.

3.9 The Simply Loaded 4-Node Lattice

For this discussion, please refer to Fig. 6. The simplifications of Section 3.2 do not apply to this lattice. Nonetheless, it is known from the figure that $_1^2x = \frac{4}{3}x = s_{1-\Delta}$ and $_2^3x = s_1$. In addition, the figure informs one that $\lambda/L = 1/(3 - 2\Delta)$. A tabulation of the nodal energies (Eq. 47) gives

$$
\begin{aligned}
2E_1 &= & e_{1-\Delta} + e_{2-\Delta} + e_{3-2\Delta} \\
2E_2 &= e_1 + e_{1-\Delta} + e_{2-\Delta} \\
2E_3 &= e_1 + e_{1-\Delta} + e_{2-\Delta} \\
2E_4 &= & e_{1-\Delta} + e_{2-\Delta} + e_{3-2\Delta}
\end{aligned}
$$

$$
E = \sum_{i=1}^{4} E_i = e_1 + 2e_{1-\Delta} + 2e_{2-\Delta} + e_{3-2\Delta} \tag{86}
$$

A direct application of Eq. 54 reveals

$$
F = \frac{1}{3 - 2\Delta}\left[f_1 \cdot 1 + 2f_{1-\Delta}(1 - \Delta) + 2f_{2-\Delta}(2 - \Delta) + f_{3-2\Delta}(3 - 2\Delta)\right] \tag{87}
$$

At the moment, Eq. 87 looks nothing like the applied lattice force derived in Section 2.7, given in 2 separate forms in Eq. 39. Those 2 forms correspond to the co-equal internal lattice force at 2 separate internal locations, given by $\hat{F}_1$ (Eq. 36) and $\hat{F}_2$ (Eq. 37). However, consider the generalized expression for average internal lattice force, $\bar{F}$, given in Eq. 4. For the 4-node simply loaded lattice under consideration, it becomes

$$
\bar{F} = \frac{2\hat{F}_1(1 - \Delta) + \hat{F}_2}{3 - 2\Delta} \quad . \tag{88}
$$

Knowing that $\bar{F} \equiv F$ for any simply loaded lattice, substituting Eqs. 36 and 37 into

this expression produces Eq. 87.

This tautological reduction implies that the external force derived from the energy approach, Eq. 87, corresponds in form and magnitude to the average internal force derived using a force approach, Eq. 39, for a 4-node, simply connected 1-D lattice.

To generate the applied boundary force, the relation Eq. 58 needs to be applied to a boundary node of the 4-node simply loaded lattice (see Fig. 6). A direct application of Eq. 58 on node 1 gives

$$F_1 = -\left[f_{1-\Delta} \cdot (-1) + f_{2-\Delta} \cdot (-1) + f_{3-2\Delta} \cdot (-1) \right] \quad , \tag{89}$$

which simplifies to

$$F_1 = f_{1-\Delta} + f_{2-\Delta} + f_{3-2\Delta} \quad , \tag{90}$$

which matches 1 of the expressions given in Eq. 39 for the external loading of the simply loaded 4-node lattice.

To generate the constraint necessary for the evaluation of Δ, the relation Eq. 58 needs to be applied to an interior node of the 4-node simply loaded lattice. A direct application of Eq. 58 on node 2 gives

$$F_2 \equiv 0 = -\left[f_{1-\Delta} \cdot (1) + f_1 \cdot (-1) + f_{2-\Delta} \cdot (-1) \right] \quad , \tag{91}$$

which simplifies to

$$f_1 + f_{2-\Delta} = f_{1-\Delta} \quad , \tag{92}$$

which is the exact constraint of Eq. 38.

3.10   The Simply Loaded 5-Node Lattice

For this discussion, please refer to Fig. 7. The simplifications of Section 3.2 do not apply to this lattice. Nonetheless, it is known from the figure that $\frac{2}{1}x = \frac{5}{4}x = s_{1-\Delta}$ and $\frac{3}{2}x = \frac{4}{3}x = s_1$. In addition, the figure informs one that $\lambda/L = 1/(4 - 2\Delta)$. A

tabulation of the nodal energies (Eq. 47) gives

$$2E_1 = \quad e_{1-\Delta} + \quad e_{2-\Delta} + e_{3-\Delta} + e_{4-2\Delta}$$

$$2E_2 = \quad e_1 + e_{1-\Delta} + e_2 + \quad e_{3-\Delta}$$

$$2E_3 = 2e_1 + \quad 2e_{2-\Delta}$$

$$2E_4 = \quad e_1 + e_{1-\Delta} + e_2 + \quad e_{3-\Delta}$$

$$2E_5 = \quad e_{1-\Delta} + \quad e_{2-\Delta} + e_{3-\Delta} + e_{4-2\Delta}$$

$$E = \sum_{i=1}^{5} E_i = 2e_1 + 2e_{1-\Delta} + e_2 + 2e_{2-\Delta} + 2e_{3-\Delta} + e_{4-2\Delta} \qquad (93)$$

A direct application of Eq. 54 reveals

$$F = \frac{1}{4 - 2\Delta}\Big[2f_1 \cdot 1 + 2f_{1-\Delta}(1 - \Delta) + f_2(2) + 2f_{2-\Delta}(2 - \Delta) \\ + 2f_{3-\Delta}(3 - \Delta) + f_{4-2\Delta}(4 - 2\Delta) \Big] \qquad (94)$$

On its face, Eq. 94 looks nothing like the applied lattice force derived in Section 2.8, given in 2 separate forms in Eq. 45. Those 2 forms correspond to the co-equal internal lattice force at 2 separate internal locations, given by $\hat{F}_1$ (Eq. 41) and $\hat{F}_2$ (Eq. 42). However, as before, consider the average internal lattice force given in Eq. 4, for this 5-node lattice:

$$\bar{F} = \frac{2\hat{F}_2 + 2\hat{F}_1(1 - \Delta)}{4 - 2\Delta} \qquad . \qquad (95)$$

Since $\bar{F} \equiv F$ for any simply loaded lattice, substituting Eqs. 41 and 42 in this expression produces Eq. 94.

This tautological reduction implies that the external force derived from the energy approach, Eq. 94, corresponds in form and magnitude to the average internal force derived using a force approach, Eq. 45, for a 5-node, simply connected 1-D lattice.

To generate the applied boundary force, the relation Eq. 58 needs to be applied to a boundary node of the 5-node, simply loaded lattice (see Fig. 7). A direct application of Eq. 58 on node 1 gives

$$F_1 = -\Big[f_{1-\Delta} \cdot (-1) + f_{2-\Delta} \cdot (-1) + f_{3-\Delta} \cdot (-1) + f_{4-2\Delta} \cdot (-1) \Big] \quad , \qquad (96)$$

which simplifies to

$$F_1 = f_{1-\Delta} + f_{2-\Delta} + f_{3-\Delta} + f_{4-2\Delta} \quad , \tag{97}$$

which matches 1 of the expressions given in Eq. 45 for the external loading of the simply loaded 4-node lattice. An evaluation of F_5 would reveal that $F_5 = -F_1$.

To generate the constraint necessary for the evaluation of Δ, the relation Eq. 58 needs to be applied to an interior node of the 5-node simply loaded lattice. A direct application of Eq. 58 on node 2 gives

$$F_2 \equiv 0 = -\left[f_{1-\Delta} \cdot (1) + f_1 \cdot (-1) + f_2 \cdot (-1) + f_{3-\Delta} \cdot (-1) \right] \quad , \tag{98}$$

which simplifies to

$$f_{1-\Delta} = f_1 + f_2 + f_{3-\Delta} \quad , \tag{99}$$

which is the exact constraint of Eq. 44. An evaluation of F_4 would reveal the identical constraint.

4. Conclusion

In this report, fully connected 1-D lattices, in which every lattice node mechanically interacts with every other lattice node through a nonlinear spring, were examined. No assumptions were made about the nature of the relationship governing the internodal spring function (except by way of example). Thus, there is no presumption of harmonic lattice behavior. Specific lattice configurations were considered, ranging from the simplest 2-node structure to a 6-node lattice. Two constraints were respectively applied to each lattice, where applicable: either the lattice was constrained to uniform internodal spacing; or else, the applied load was restricted to simple loading at the lattice boundary.

That a uniformly spaced lattice constraint is incompatible with simple boundary loading may not be surprising; however, nor is it intuitive. To maintain uniform spacing in the lattice, external forces must be applied even to internal nodes of the lattice. In contrast, a simple boundary load upon the lattice produces a deformation which is non-uniform. This occurs because the aggregate spring constant is lesser for nodes near the lattice boundary, because of fewer nodal neighbors. Reconciling these discrepancies is essential to understanding the lattice equilibrium problem.

Two methodologies were applied to each lattice configuration that was considered:

1) a force approach, in which free-body diagrams were developed that conceptually cut the lattice at various locations in its internal structure, to establish and resolve the equations of force equilibrium; and

2) an energy approach, in which the potential energy of every internodal interaction was tabulated. With this approach, both the global as well as the individual nodal force responses are derived quantities from the total potential energy.

With both approaches, relationships were established between the local internodal (internal) forces and the (external) macroscopic forces applied to the body. The goal of the report was to confirm the expected, that these 2 approaches yielded compatible results. That confirmation was achieved for all configurations examined.

Each approach has its strong and weak aspects. While the force approach may be conceptually easier to understand, the execution of it proves more tedious. In addition, while the current effort was limited to 1-D lattices, the force approach will need to further rely on vector mechanics as the lattice dimension extends to 2-D or 3-D, thereby adding increasing complexity. The energy approach does not suffer this drawback. The ability to generalize solutions also proves much easier with the energy approach, as was seen in Section 3.8.

Key to achieving the proper formulation with the energy approach is a proper use of the chain rule for differentiation, which takes the forms of Eqs. 51 and 56, as applied to the 1-D lattice compression problem. When utilized to differentiate the system energy with respect to the body dimension, the resulting global force equation, Eq. 54, exhibits a multiplier on each internodal force that is proportional to the range of the interaction. This particular multiplier is peculiar to uniform lattice expansions/contractions (as opposed to distortions from wave propagation), in that distal nodes converge a greater distance than proximate nodes under a uniform decrease in lattice spacing.

An interesting result of the study was the revelation that the global force F that derives from the energy approach was not necessarily equal to the net force F_{net} applied to the lattice. Rather, F was equivalent, for all boundary conditions considered, to the average internal force $\bar{F}$ as integrated throughout the lattice. The discrepancy can arise, since a uniformly spaced lattice (for $N > 3$) receives applied

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loads to internal nodes, which means that such loads do not deform the complete lattice structure, but only parts thereof. However, as shown in section 3.8, the discrepancy will tend to 0 as the lattice size N grows, assuming the internodal lattice stiffness decreases with increasing nodal separation, as in the case of atomic lattices.

5. References

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List of Symbols, Abbreviations, and Acronyms

λ – the nominal lattice spacing under the given load (*i.e.*, the actual lattice spacing "far" from the lattice boundary).

Δ – the relative amount by which a given lattice cell length may deviate from the nominal value of λ, owing to a change in lattice stiffness near the boundary of the lattice. Its value is limited to real numbers in the range $\Delta < 1$.

ϵ_i – the infinitesimal fraction of λ by which a lattice node is moved in order to assess a change in the net potential energy of the lattice.

N – the total number of nodes in the 1-D lattice.

L – the total length of the macroscopic lattice, $L \approx (N-1)\lambda$. However, when the lattice is constrained to uniform spacing, the approximation is exact.

s_k – shorthand for internodal distance in wavelengths, such that $s_k \equiv k\lambda$. The index k may take on non-integer values if the lattice spacing varies from the uniformly spaced configuration.

$_i^m x$ – the (positive) physical separation distance between node pair $i \curlyvee m$ of the lattice. For the special case of a uniformly spaced lattice, $_i^m x \equiv s_{|m-i|} \equiv |m-i|\lambda$. Whereas the notation $_i^m x$ provides the specificity of both nodes in the node pair, the notation s_k allows the convenience of tracking only the separation distance in node pair $i \curlyvee m$, without the requirement to know the values of i and m, specifically.

f_k – the internal spring force, using the shorthand that $f_k \equiv f(s_k) \equiv f(k\lambda)$ (note that $f_0 \equiv 0$). The index k may take on non-integer values.

F_i – the external Cartesian load applied to node i. Positive denotes a force in the rightward direction, while negative denotes a force to the leftward direction. Note that, because of the symmetric equilibrium being imposed, $F_j = -F_{N+1-j}$.

F_{net} – the net externally applied load, $F_{\text{net}} = \sum_{j=1}^{N/2} F_j$.

$\hat{F}$ – when subject to simple boundary-loading, the internal force at any point in the lattice.

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$\hat{F}_j$ – the internal force in a lattice (positive implies compression) that exists *between* nodes j and $(j+1)$ (*e.g.*, $\hat{F}_1$ is the internal force between the node pair $1 \vee 2$). Note that, because of the symmetric equilibrium being imposed, $\hat{F}_j = \hat{F}_{N-j}$. When a lattice is subject to simple boundary loading, $\hat{F}_j \equiv \hat{F}$.

$\bar{F}$ – the average internal load in the lattice, given generally by

$$\bar{F} = \frac{1}{L} \int_0^L \hat{F}(x)\, dx$$

$$= \sum_{j=1}^{N-1} \hat{F}_j \,^{j+1}_{j}x \Bigg/ \sum_{j=1}^{N-1} \,^{j+1}_{j}x \quad .$$

In the case of a uniformly spaced lattice, where $^{j+1}_{j}x = \lambda$, the above reduces to

$$\bar{F} = \frac{1}{N-1} \sum_{j=1}^{N-1} \hat{F}_j \quad .$$

F – when using the energy method, F is the derivative of the system mechanical energy with respect to a change in the body dimension. It was determined that, in the case of both uniformly spaced as well as simply loaded lattices, F is equivalent to the average internal force, $\bar{F}$.

e_k – an atom's potential energy arising from an interaction with an atom at a distance of s_k. Thus, $e_k \equiv e(k\lambda)$ Note that $e_0 \equiv 0$. The index k may take on non-integer values.

E_i – Net energy potential associated with atom i in the lattice.

E – the total energy potential of the lattice, $E = \sum_{i=1}^{N} E_i$.

$i_1 \vee i_2$ – the node pair i_1, i_2 through which an internal nonlinear spring force acts.

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  J CONDON
RDRL WML H
  T EHLERS
  T FARRAND
  E KENNEDY
  L MAGNESS
  C MEYER
  B SORENSEN
  R SUMMERS
RDRL WMM B
  B LOVE
  G GAZONAS
RDRL WMM E
  P PATEL
RDRL WMP
  D LYON
  S SCHOENFELD
  T VONG
RDRL WMP A
  P BERNING
  W C UHLIG
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  C HOPPEL
  S SATAPATHY
RDRL WMP C
  T BJERKE
  R BECKER
  S R BILYK
  D CASEM
  J CLAYTON
  D DANDEKAR
  M J GRAHAM
  M GREENFIELD
  B LEAVY
  S SEGLETES
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  R FREY
  M KEELE
  D KLEPONIS
  H W MEYER
  R MUDD
  S SCHRAML
  B SCOTT
  K STOFFEL
  V WAGONER
  W WALTERS
  M ZELLNER
RDRL WMP E
  P SWOBODA
  P BARTKOWSKI

  M BURKINS
  D HACKBARTH
  D HORNBAKER
  E HORWATH
  J HOUSKAMP
  C KRAUTHAUSER
RDRL WMP F
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  R BITTING
  C CUMMINS
  E FIORAVANTE
  D FOX
  R GUPTA
  S HUG
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  R EHLERS
  S KUKUCK
  C PECORA